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Vortex Sheet Approximation of Boundary Layers

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A grid free method for approximating incompressible boundary layers is introduced. The computational elements are segments of vortex sheets. The method is related to the earlier vortex method; simplicity is achieved at the cost of replacing the Navier-Stokes equations by the Prandtl boundary layer equations. A new method for generating vorticity at boundaries is also presented; it can be used with the earlier vortex method. The applications presented include (i) flat plate problems, and (ii) a flow problem in a model cylinder–piston assembly, where the new method is used near walls and an improved version of the random choice method is used in the interior. One of the attractive features of the new method is the ease with which it can be incorporated into hybrid algorithms.

INTRODUCTION

Some time ago we introduced a random vortex method for solving the Navier-Stokes equations [4]. The idea of the method was to approximate Euler's equations by analyzing the interaction of vortices, and then introduce the effects of viscosity by adding to the motion of the vortices an appropriate random component. This method has been further developed by, among others, Ashurst [1], Leonard [14, 15], Meng [19], Rogallo [20], and Shestakov [23, 24], and theoretical analyses have been carried out by Marsden *et al.* [17–8], among others. One attractive feature of the method is the fact that the tangential boundary condition is satisfied through vorticity creation, a procedure which mimics an essential physical phenomenon (see [2, 16, 25]).

That method has of course not solved all the outstanding problems of high Reynolds number flow. Some of the difficulties in its use have been: (i) the rate of convergence near boundaries has been slow, and as a result it is not always easy to ensure that the results obtained are independent of numerical parameters except possibly when points of separation can be determined a priori; (this point has been investitigated by Ashurst [1] and Rogallo [26]); (ii) the dependence of the method on an assumed structure of the vortices makes analysis difficult, in particular in the three dimensional case (see e.g. Leonard [14], [15]); (iii) in interior flow problems, the cost of the calculation can be substantial (see e.g. Ashurst [1]); Shestakov [23], [24] has derived a hybrid method which partly overcomes this difficulty. In the present paper, we present a new vorticity

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generation method which should overcome problem (i) above, and introduce a related vortex method which solves the Prandtl boundary layer equations; in this method the vortex interaction is not singular, problem (ii) disappears, and the method can be used near boundaries in hybrid methods. A more general (but much more complicated) vortex method for the analysis of three dimensional turbulent boundary layers will be described elsewhere [7].

Principle of the Method

The boundary layer equations can be written in the form (see e.g. Schlichting [21])

$$\partial_t \xi + (\mathbf{u} \cdot \nabla)\, \xi = \nu \partial_y{}^2 \xi, \tag{1a}$$

$$\xi = -\partial_y u, \tag{1b}$$

$$\partial_x u + \partial_y v = 0, \tag{1c}$$

where $\mathbf{u} = (u, v)$ is the velocity, u is tangential to the boundary and v is normal to the boundary, x is the spatial coordinate tangential to the boundary and y is the coordinate normal to the boundary, ξ is the vorticity and ν is the viscosity. We assume the wall is at $y = 0$ and the fluid fills out the half-space $y \geqslant 0$. The boundary conditions are

$$\mathbf{u} = 0 \text{ at } y = 0, \tag{2a}$$

$$u(x, y = \infty) = U_\infty(x). \tag{2b}$$

Additional conditions may be needed on the left and/or on the right. Equation (1c) can be integrated in the form

$$v(y) = -\partial_x \int_0^y u(x, z)\, dz \tag{3a}$$

and Eq. (1b) yields

$$u(x, y) = U_\infty - \int_y^\infty \xi(x, z)\, dz. \tag{3b}$$

It can be readily seen that if ξ is known, (3a) and (3b) yield u and v.

Consider a collection of N segments S_i of vortex sheets, of intensities ξ_i, $i = 1,..., N$ (i.e., segments of a straight line such that u on one side of S_i and u on the other side of S_i differ by ξ_i). S_i is parallel to the x axis, of length h and center $\mathbf{x}_i = (x_i, y_i)$. The x component of the velocity of S_i due to the presence of the other segments can be found from (3b), which yields approximately

$$u_i = U_\infty(x_i) - \tfrac{1}{2}\xi_i - \sum_j \xi_j d_j \tag{4a}$$

where

$$d_j = 1 - (| x_i - x_j |/h) \tag{4b}$$

and the sum $\sum$ is over all S_j such that

$$y_j > y_i \text{ and } | x_i - x_j | < h \tag{4c}$$

(i.e., $0 \leqslant d_j \leqslant 1$).

The vertical velocity v_i of S_i can be approximated from (3a) by

$$v_i = -(I_1 - I_2)/h \tag{5a}$$

where I_1 and I_2 approximate respectively $\int_0^{y_i} u(x + h/2, y)\, dy$ and $\int_0^{y_i} u(x - h/2, y)\, dy$, and can be taken as

$$I_1 = U_\infty(x_i + h/2)\, y_i - \sum_{j^+} \xi_j d^+{}_j y^*{}_j, \tag{5b}$$

$$I_2 = U_\infty(x_i - h/2)\, y_i - \sum_{j^-} \xi_j d^-{}_j y^*{}_j, \tag{5c}$$

where

$$d^+{}_j = 1 - | x_i + h/2 - x_j |/h, \tag{5d}$$

$$d^-{}_j = 1 - | x_i - h/2 - x_j |/h, \tag{5e}$$

$$y^*{}_j = \min(y_i, y_j), \tag{5f}$$

the sum $\sum_+$ is over all S_j such that $0 \leqslant d^+{}_j \leqslant 1$ and the sum $\sum_-$ is over all S_j such that $0 \leqslant d^-{}_j \leqslant 1$. Note that the total number of interactions between vortex sheets is small, in particular in comparison with what happens when point vortex interactions are taken into account; one can use sorting algorithms to minimize the number of decisions involved in carrying out the several summations.

Thus, the motion of vorticity described by the equations

$$\partial_t \xi + (\mathbf{u} \cdot \nabla)\, \xi = 0,$$

$$\partial_x u + \partial_y v = 0,$$

$$\xi = -\partial_y u$$

can be approximated by

$$x_i^{n+1} = x_i^n + k u_i,$$

$$y_i^{n+1} = y_i^n + k v_i,$$

where k is a time step, and $x_i^n \equiv x_i(nk)$, $y_i^n \equiv y_i(nk)$. The effect of viscosity can then be included by adding to the deterministic formula for y_i^{n+1} a random variable η_i drawn from a gaussian distribution with mean 0 and variance $2\nu k$; this yields the algorithm

$$x_i^{n+1} = x_i^n + ku_i, \tag{6a}$$

$$y_i^{n+1} = y_i^n + kv_i + \eta_i. \tag{6b}$$

The several values of η_i are independent, u_i is given by (4) and v_i by (5). The boundary conditions $u = U_\infty$ at $y = \infty$ and $v = 0$ at $y = 0$ are automatically satisfied; the boundary condition $u = 0$ at $y = 0$ will be satisfied through a vorticity creation operation described in the next section. The statistical error in Eq. (6) can be reduced by a tagging method which will also be described below. Note that no grid is introduced; there is no lower bound to the thickness of the boundary layer which can be resolved, and no differencing occurs across the layer. Furthermore, the solution is computed in the (x, y) plane, without a change of variables, and thus it should be easy to match the computed boundary layer solution with an inviscid solution outside the layer.

VORTICITY CREATION

In [4] we proposed the following algorithm for satisfying the tangential boundary condition on $\mathbf{u}$: Let $\mathbf{u}_0$ be the flow satisfying the equation of motion and the boundary condition $v_0 = 0$. If at the wall $u_0 \neq 0$, the effect of viscosity will be to create a thin boundary layer near the wall; the total vorticity in the layer per unit length of the wall is

$$\int_{\text{wall}}^{\text{interior}} \xi \, dy = \int_{\text{wall}}^{\text{interior}} \frac{\partial u}{\partial y} \, dy = u_0$$

i.e., one has to create a vortex sheet of strength u_0 per unit length of the wall; this vortex sheet is then broken up into elements and allowed to participate in the subsequent motion of the fluid. The vorticity elements which cross the wall are lost; their vorticity will of course be recreated at the next step. This construction was offered in [4] on heuristic and physical grounds.

To understand the nature of the approximations made, it is adequate to consider the diffusive part of the equation, i.e., the diffusion equation $\partial_t \xi = \nu \partial_y^2 \xi$ with the boundary $u = 0$. The gaussian random variable provides an approximate solution of the whole space heat equation (since the Green's function of the heat equation in the absence of boundaries is a gaussian function). The subsequent deletion of the vortices which cross the boundary and the creation of a vortex sheet of intensity u serve to project the solution of the whole space heat equation on the subspace of functions which vanish outside the domain of integration. This formulation is due to Marsden and McCracken; its convergence as $k \to 0$ in the case of linear equations such as the heat equation follows from the work of Kato [11, 12] (see [8] for a review). It has, however, been observed, computationally by Ashurst [1], Rogallo [20] and the author, and theoretically in [8], that the rate of convergence near the wall as $k \to 0$ is slow, in particular since the boundary condition $u = 0$ on the wall is satisfied only in the limit. We therefore introduce an alternative to the earlier approximation in which the boundary condition is satisfied exactly except possibly at a finite set of values of t. The velocity field is extended across the wall at the beginning of each time step by the anti-symmetry $u(x, -y) = -u(x, y)$, where the wall is assumed to be at $y = 0$. As a result, $\xi(x, -y) = \xi(x, y)$ for $y \neq 0$, and a vortex sheet appears at $y = 0$ if the tangential velocity does not vanish at the wall. This anti-symmetry replaces the vorticity creation operation used in earlier work. The whole space diffusion equation is then solved by a random walk, for a time k. using as initial data the extended solution. Algorithmically, this is equivalent to (i) creating a vortex sheet of strength $2u_0$ per unit length of the wall, and (ii) bouncing those vortices which cross the wall back into the fluid, i.e., if at the end of a time step a vortex finds itself at (x_i, y_i), $y_i < 0$, it is returned to $(x_i, -y_i)$. For some analysis, see [8].

Thus, we take points $Q_1, \ldots, Q_m$ at the wall, such that the distances $\overline{Q_1Q_2}$, $\overline{Q_2Q_3}$ equal h. At each point Q_i we evaluate the tangential velocity u_0, using the obvious specialization of Eqs. (4). We imagine then a vortex sheet of strength $2u_0$ at Q_i. In order to have a reasonable approximation of the diffusion equation at a later time, we create at Q_i not a simple vortex sheet, but some number l of sheets such that the intensity of each is less in absolute value than a predetermined $\xi_{\max}$. At the next step, these sheets will behave according to the laws (6). Some obvious programming precautions must be taken: the vortex sheets which have just been created and are taking their first random step may jump out of the domain of integration; these should be lost and not bounced back (or else the wall symmetry will be violated). One must also ensure that the term $\frac{1}{2}\xi_i$ in the formula (4a) does not add an unnecessary horizontal component to the motion of the newly created sheets.

A substantial reduction in the statistical error can be made by observing that in Eq. (1) diffusion takes place only in the y direction; thus the numbers η_i used in (6) need be independent of each other only when they are used with vortex segments whose centers lie in a narrow strip perpendicular to the wall. This fact can be used in the following way: As vortices are created, they are assigned integer tags, m_i being the tag assigned to the ith vortex sheet element. At each time step, a tag not used before is chosen and assigned to one vortex element at each boundary point Q_j at which at least one element is created. A second tag is then chosen, and assigned to one element at each point where at least two elements are created, etc. The effect of the tagging is to piece together the elements created at the several boundary points into coherent vortex sheets, with the elements of each sheet identified by a common tag. When the random numbers η_i are chosen for use in (6), all elements with the same tag are assigned the same η. This is the variance reduction procedure. In parallel flows, its effect is to make the sums in (5) identically zero (and thus reduce their variance to zero).

Flow Past a Semi-infinite Flat Plate

Consider a semi-infinite flat plate placed on the positive x axis, with a fluid of density 1 occupying the half plane $y > 0$. At time $t = 0$ the fluid is impulsively set in motion with velocity $U_\infty = 1$. We shall apply our method to the analysis of this problem, with the aim of comparing the results with the well known solution (see, e.g., Schlichting [21]).

The leading edge singularity presents no difficulty. One fairly minor detail requires some attention: we are going to compute over a finite length of the plate, say for $0 \leqslant x \leqslant a$. From Eqs. (1) it follows that no boundary condition need, or indeed may, be imposed at $x = a$, since the flow of information will be to the right only. However, formula (5a) is essentially a centered difference approximation to $\partial_x \int u\, dy$, and may give rise to a spurious flow of information to the left. This is easily corrected by removing all vortex sheets which cross $x = a$ and by not allowing those sheets whose centers lie between a and $a - 2h$ to have any motion in the y direction — they are thus merely convected downstream without disturbing the flow to their left.

The numerical parameters at our disposal are h, k, and $\xi_{\max}$. The method is unconditionally stable, and h, k are constrained only by an accuracy requirement $uk \leqslant O(h)$. Convergence should occur as h, k, $\xi_{\max}$ all tend to zero. As these parameters decrease, the number N of sheets in the calculation increases, the amount of labor increases, but both the differencing error in (5) and the statistical error decrease.

The calculations were pursued until a steady state had been reached and maintained for a while. In a steady state, the drag D on the portion of the plate between O and a point X can be evaluated from the momentum defect formula ([21, p. 161])

$$D = \int_0^\infty u(U_\infty - u)\, dy, \qquad u = u(X, y).$$

The integral can be evaluated as follows: Consider all the vortex sheets S_i, $i = 1, 2,..., M$ whose centers satisfy $|x_i - X| < h$. Assume that they are numbered in such a way that $y_1 \leqslant y_2 \leqslant y_3 \leqslant \cdots \leqslant y_{12}$. Then we have approximately

$$D = \sum_{i=1}^{M} u_i(U_\infty - u_i)\, \Delta y_i\,,$$

where, as before,

$$u_i = U_\infty(X) - \tfrac{1}{2}\xi_i - \sum_{j=i+1}^{M} \xi_j d_j\,,$$

with $d_j = |x_j - X|/h$, $\Delta y_i = y_i - y_{i-1}$, $y_0 = 0$.

Define the streamwise Reynolds number

$$R = U_\infty X/\nu;$$

to first order in $R^{-1/2}$ we have from boundary layer theory

$$D = 0.664/R^{1/2}. \tag{7}$$

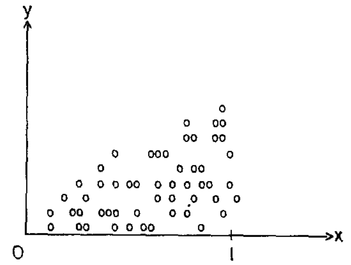

FIG. 1. Vortex sheets over a flat plate.

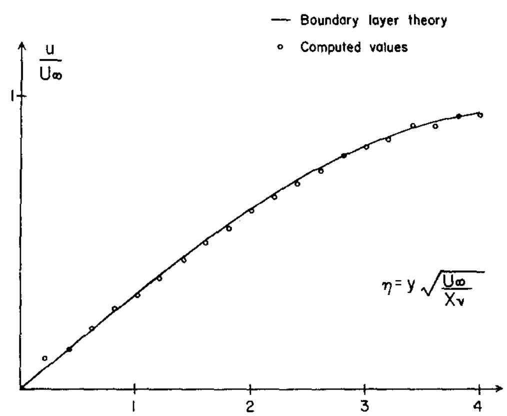

FIG. 2. Horizontal velocity in Blasius flow.

In Fig. 1 we display a typical vorticity configuration: an O corresponds to the center of a vortex sheet. This configuration was obtained with $k = 0.2$, $h = 0.2$, $\xi_{\max} = 0.1$, $\nu = 10^{-6}$, at $t = 5.0$.

We found experimentally that for $k \leqslant 0.2$, $h \leqslant 0.2$, $\xi_{\max} \leqslant 0.1$ the statistical error dominated all others; this error decreases rather slowly as the number of vortex sheets increases, but will not be particularly troublesome in later applications (see below). One method for reducing the statistical error in the steady state is to average the solution over a number of time steps (see [23, 24]). In Fig. 2 we display the velocity profile averaged over 20 steps with $\nu = 10^{-6}$, $k = 0.2$, $h = 0.2$, $\xi_{\max} = 0.1$, $8 \leqslant t \leqslant 12$, compared with the analytic boundary layer solution.

The drag computed at $\nu = 10^{-6}$, averaged over 20 steps, is 6.69×10^{-2}, compared with the value 6.64×10^{-2} obtained from (7). If one considers the successive values of D at the several time steps to be successive estimates of D, then the standard deviation of the computed answer is 0.4×10^{-2}. At $\nu = 10^{-4}$, the computed value of D is 0.669, with standard deviation .04, compared to the value $D = 0.664$ obtained from (7). In all our calculations, the averages of the computed D converged to the mean value much faster than one would have expected from the estimates of the standard deviation. No explanation is offered, and we do not know how general this effect may be. The typical number of vortex sheets in these runs is 100, and a typical running time is 20 seconds on the CDC 6400 computer at Berkeley.

I also ran some problems where U_∞ was not constant but had the form

$$U_\infty = 1 - A \sin \pi x.$$

TABLE 1

Separation and Reattachment

η	x: 0.1	0.2	0.3	0.4	0.5	0.6
0.04	0.34	0.15	0.00	−0.12	−0.04	0.04
0.08	0.38	0.17	0.02	−0.10	−0.03	0.05
0.12	0.39	0.18	0.03	−0.09	−0.01	0.06
0.16	0.39	0.19	0.04	−0.07	0.00	0.09
0.20	0.41	0.21	0.05	−0.06	0.01	0.09
0.24	0.42	0.22	0.06	−0.05	0.02	0.10
0.28	0.44	0.23	0.07	−0.05	0.02	0.11
0.32	0.44	0.24	0.08	−0.03	0.04	0.11
0.36	0.45	0.24	0.10	−0.02	0.03	0.12
0.40	0.46	0.26	0.10	−0.02	0.04	0.13
0.44	0.46	0.27	0.11	−0.00	0.05	0.13
0.48	0.46	0.28	0.12	0.01	0.06	0.12
0.52	0.47	0.28	0.14	0.03	0.07	0.12

For large enouch A one expects separation, reattachment and a recirculation bubble. However, in a direct method (as opposed to an inverse method [13]) one expects the singularities at separation and reattachment to taint the solution (see [22, 9]). In Table I we display some values of u obtained with $k = .1$, $h = .1$, $\xi_{\max} = 1$, $\nu = 10^{-6}$, $A = .2$, averaged over 20 steps between $t = 8$ and $t = 12$. $\eta = y/(X\nu)^{1/2}$ is the usual similarity variable. The negative values of u represent the recirculation region. A steady state was never achieved; the validity of the solution is unclear and it is in fact doubtful. No connection with the inverse method of Klineberg and Steger [13] was established. The effect of the singularities is widely believed to be removable by coupling the boundary layer calculation to the outer calculation; a method for doing this for another problem is described in the following section.

A Hybrid Algorithm Involving the Random Choice Method

We now present a hybrid algorithm in which the method described above is used near the boundaries while a different method is used in the interior of the domain. The two components of the algorithm are coupled, with the vortex sheet method serving as vorticity source for the interior method. An earlier hybrid method was presented by Shestakov [23, 24]; in Shestakov's work, a vortex blob method was used near the walls, and a difference method was used in the interior, with a coupling based on a careful use of spline interpolation. A hybrid method based on the use of vortex sheets near walls and vortex monopoles in the interior will be presented elsewhere [7].

Here we use as an interior method a version of the random choice method for compressible flow [5, 6]. Thus, not only do we use different methods in the interior and near walls, but we also make different assumptions about compressibility: We have viscous incompressible flow near the walls and inviscid compressible flow in the interior. There are two sets of reasons for doing this:

(a) Difficulties with interior viscosity. One may well believe that the numerical viscosity associated with finite difference or finite element methods has little effect as long as one stays away from walls, but it is not clear what "staying away from walls" should mean. The interior method must reach quite close to the walls, and earlier numerical experiments [23] indicate that unless the interior viscosity is tightly controlled, e.g., through the use of a very fine grid, the results may be substantially in error. The random choice method has effectively no numerical viscosity and is available for use. Since all we want to do is demonstrate how the sheet method can be coupled to an interior method, the random choice method is acceptable, as long as the Mach number near the walls is reasonably small.

(b) Ulterior motives. The methods of this paper will be used on the analysis of reacting gas flow, and in that context it is believed that the particular mixture of methods we use here will be most appropriate.

The most important problem is to find a reasonable way for coupling the interior and the boundary. If it is known in advance that the boundary layer will not separate, this is trivial, since all one has to do is use tangential velocities from the interior as velocities at infinity for the boundary. In interesting cases it is, however, essential that the layer act on the interior as well, since it may have a crucial impact on the interior flow, and since some boundary–interior interaction is needed to counteract the separation singularity. In the examples described in the following section we proceeded as follows: used the tangential velocity at the wall of the interior calculation as velocity at infinity for the boundary layer calculation, and impressed upon the interior calculation the velocity normal to the wall induced by the boundary layer calculation.

This last normal velocity was computed as follows: let P be a point at the wall, with coordinates $((l + \frac{1}{2}) \Delta, 0)$, where l is an integer and Δ is the grid size in the interior. The momentum lost due to the boundary layer above P can be approximated by

$$U_{l+1/2} = \sum \xi_j d_j y_j$$

where $d_j = |\, x_j - (l + \frac{1}{2})\, \Delta \,|/h$, (x_j, y_j) is the center of the vortex sheet S_j with vorticity ξ_j, and the sum is over all S_j such that $0 \leqslant d_j \leqslant 1$ (see Eqs. (5)). Then the normal velocity at $x = l\Delta$ is approximately $(U_{l+1/2} - U_{l-1/2})/\Delta$. This velocity is imposed on the interior calculation at the boundary.

The programming details of the joint vortex sheet-random choice calculation require a somewhat lengthy explanation, mostly because of the relative complexity of the random choice program. The equations solved in the interior are the usual Euler equations. As described in [5], one full step of the random choice method for these equations consists of four quarter steps of length $k/2$. Let $\mathbf{V}_{i,j}^n \equiv \mathbf{V}(i\Delta, j\Delta, nk)$ denote the solution vector. At the beginning of the step we have $\mathbf{V}_{i,j}^n$ for i, j integers. In the first quarter step we compute $\mathbf{V}_{i+1/2,j}^{n,1/4}$, in the second quarter step we compute $\mathbf{V}_{i+1/2,j+1/2}^{n,1/2}$, in the third quarter step we compute $\mathbf{V}_{i,j+1/2}^{n,3/4}$, and in the last quarter step we compute $\mathbf{V}_{i,j}^{n+1}$. To obtain one new value for the vector $\mathbf{V}$ at a point one solves a Riemann problem, which is them sampled. The sampling strategies have been described in some detail in [6]; they involve "random" numbers θ. A Riemann problem is an initial value problem for the equations of motion in which the initial data are discontinuous. Its solution contains a slip line; i.e., a line which divides the fluid initially to the left of the discontinuity from the fluid initially to the right of the discontinuity.

Near the boundary, symmetry conditions can be used to formulate the appropriate Riemann problems. In the program used here, which is a refinement of the earlier program [5], the physical domain is not always fixed with respect to the computational grid. All points are identified by an integer tag q, with $q = 1$ for points in the interior of the domain and $q = 0$ for points outside the domain. q is treated as a passive quantity and propagates as part of the calculation, depending on the relative position of the slip line and the sampling point. If $q = 0$ for both initial states in the Riemann problem no calculation need be carried out. If $q = 1$ for both states we have an interior point, and if we have two distinct values of q the boundary symmetry conditions are applied. As already partly described in [6], if the "random" numbers θ are picked so that the first two are $\geqslant 0$, the next two $\leqslant 0$, etc., and if the bottom and left boundaries coincide with lines $x = I\Delta$, $y = J\Delta$, I, J integers, while the top and right boundaries coincide with lines $x = (I' + \frac{1}{2})\,\Delta$, $y = (J' + \frac{1}{2})\,\Delta$, I', J' integers, then stationary boundaries remain stationary on the grid. If the boundaries are chosen as we have just described, then one boundary layer calculation step must be made every two interior quarter steps, and the conditions at infinity for the boundary layer calculation can be updated and the normal velocity imposed on the interior only once every four quarter steps (i.e., once per whole interior step), the updating occurring whenever appropriate boundary data from the interior calculation are available. It should be obvious that the fact that the random choice method does not smear out vortex sheets immediately is helpful to the success of the method.

The accuracy of the method has not yet been discussed. Clearly, our matching procedure is based on the assumption that the boundary layer thickness is at most comparable with Δ, i.e., $\Delta \geqslant O(R^{-1/2})$, where R is a Reynolds number based on an interior length scale and velocity. The accuracy of the interior Glimm method is at best $O(\Delta)$ (see [4]). Thus, the over-all accuracy is at best $O(\Delta) + O(R^{-1/2})$. This is not a surprising estimate (see, e.g., [4] for a discussion), and if it can be shown to be realistic and to hold uniformly in $R^{-1/2}$, it would represent a substantial achievement. There are of course no problems with stability, since each component of the hybrid method is unconditionally stable.

Application to Two Dimensional Flow behind a Piston

We now present an application of the preceding algorithm, an application for which the random choice interior method is well suited. We do so with words of caution. The belief that our method can handle properly the separation of a boundary layer is based more on hope than on hard analysis. The accuracy of the results is difficult to gauge through the inevitable statistical error. There are no reliable data for comparison. The best that can be said is that the results are plausible, consistent with earlier work on similar problems (see, e.g., Bernard [3]), and consistent also with the belief that the effective diffusion of the scheme equals the nominal diffusion (i.e., that the computational results correspond to the Reynolds number explicitly imposed on the calculation and not to a numerical Reynolds number intrinsic to the method).

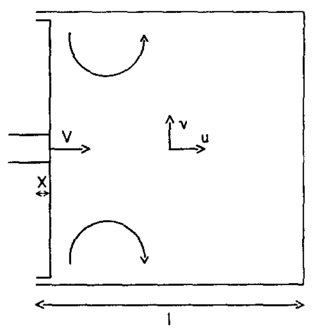

Fig. 3. Piston-cylinder flow configuration.

The flow configuration is shown in Fig. 3. A piston is pushed with velocity V into a chamber filled with gas. The initial density of the gas is $\rho = 1$, the initial pressure is $p = 1$, and the gas is initially at rest. The gas is assumed to be perfect, i.e., the internal energy is given by $\epsilon = (p/\rho)/(\gamma - 1)$, where $\gamma = 1.4$. The sound speed is $c = (\gamma p/\rho)^{1/2}$. The viscosity ν is measured in units in which $(p/\rho)^{1/2} = 1$ and the initial length of the chamber is 1. The width of the chamber is 1. Thus, the Reynolds number based on the velocity at infinity seen by the boundary layer and on the length of the chamber is

$R_V = V/\nu$. Care is taken to ensure that the Mach number $V/c \ll 1$. $V = 0$ for $t \leqslant 0$, and assumes a constant value for $t > 0$. The displacement of the piston is $X = Vt$.

In the absence of viscosity we would have a shock wave propagating into the gas, reflected at the far end, and then bouncing back and forth between the piston and the back wall. The random choice method would compute this flow with infinite resolution (see the analysis in [6]). Call this flow $\mathbf{u}_0 = (u_0, 0)$.

The effect of $\nu \neq 0$ is to superimpose on u_0 a rotational flow with the general pattern depicted in Fig. 3. The boundary layers on the top and bottom are slowed down and deflect some fluid at the piston towards the interior of the domain (see, e.g., [3]). We exhibit a calculation made with $\nu = 10^{-3}$. This relatively high value of ν is picked because the rotational effect we wish to exhibit decreases with ν. It is clear that as ν decreases our method does not break down. This ν is as large as we could pick and still observe the constraint $\Delta = O(\nu t)$. The results below must be considered while keeping in mind (i) the built-in fluctuations of the random choice method, (ii) the fact that the edge of the calculation is the edge of the boundary layer and not the boundary of the domain, and (iii) the coarseness of the interior grid.

The following parameters were used: In the interior, $\Delta = 1/13$, $k/\Delta = 0.6$, $m_1 = 7$, $m_2 = 3$ (these integers are used in the generation of the numbers θ which define the algorithm; see [5]). In the boundary layer, $h = 2\Delta = 2/13$ and $\xi_{\max} = V/5$. The

TABLE II

Horizontal Velocity behind a Piston ($t = 1.846$)

	x								
y	5/13	6/13	7/13	8/13	9/13	10/13	11/13	12/13	1
0	0.20	0.12	0.13	0.21	0.18	0.20	0.20	0.22	0.03
1/13	0.20	0.20	0.26	0.22	0.17	0.22	0.18	0.17	−0.01
2/13	0.20	0.18	0.18	0.27	0.24	0.20	0.20	0.21	0.03
3/13	0.20	0.25	0.22	0.21	0.23	0.20	0.19	0.17	0.01
4/13	0.20	0.22	0.20	0.20	0.19	0.19	0.19	0.17	0.00
5/13	.20	0.20	0.20	0.22	0.22	0.20	0.19	0.20	0.02
6/13	0.20	0.22	0.23	0.20	0.18	0.19	0.19	0.19	0.02
7/13	0.20	0.22	0.22	0.21	0.19	0.19	0.19	0.19	0.00
8/13	0.20	0.19	0.20	0.19	0.20	0.19	0.19	0.21	0.00
9/13	0.20	0.21	0.19	0.18	0.18	0.18	0.18	0.19	−0.00
10/13	0.20	0.23	0.24	0.22	0.23	0.21	0.17	0.17	0.00
11/13	0.20	0.17	0.21	0.25	0.18	0.16	0.19	0.18	−0.05
12/13	0.20	0.19	0.23	0.19	0.27	0.25	0.19	0.13	−0.03
1	0.20	0.21	0.16	0.21	0.21	0.22	0.22	0.04	−0.03

TABLE III

Vertical Velocity behind a Piston (t = 1.846)

	x								
y	5/13	6/13	7/13	8/13	9/13	10/13	11/13	12/13	1
0	0.04	−0.04	0.01	−0.02	0.03	0.04	0.01	0.01	−0.01
1/13	−0.12	−0.09	−0.00	0.03	−0.01	0.03	0.06	0.04	0.01
2/13	0.02	−0.04	−0.03	0.02	.01	0.00	0.01	0.03	0.01
3/13	0.02	0.00	0.01	0.01	−0.02	0.02	0.02	−0.00	0.00
4/13	−0.04	−0.04	−0.04	−0.03	−0.01	−0.00	0.01	0.01	−0.01
5/13	0.03	0.03	0.04	0.03	0.02	0.00	−0.01	−0.04	−0.03
6/13	−0.01	−0.03	−0.04	−0.03	−0.01	0.01	0.01	0.01	0.02
7/13	−0.01	−0.01	−0.01	0.01	0.03	0.02	0.02	0.02	0.00
8/13	0.01	0.01	−0.01	0.00	0.01	0.01	0.01	0.01	0.02
9/13	0.04	0.01	−0.01	0.02	0.00	0.02	−0.00	0.04	0.03
10/13	0.01	−0.02	−0.01	−0.01	0.02	−0.03	−0.02	−0.02	−0.02
11/13	0.08	0.01	−0.03	0.02	0.00	−0.02	−0.01	−0.03	−0.01
12/13	0.12	−0.01	−0.07	0.03	−0.01	−0.05	−0.07	−0.07	0.01
1	0.13	−0.03	−0.09	0.02	−0.04	−0.06	−0.03	−0.01	−0.02

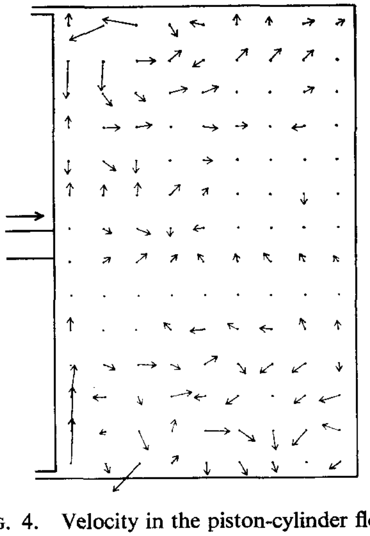

FIG. 4. Velocity in the piston-cylinder flow.

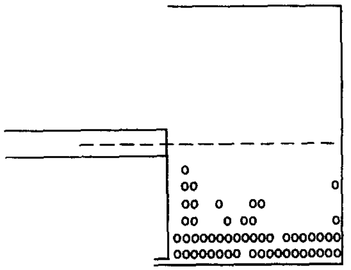

FIG. 5. Vorticity in the boundary layer of the piston-cylinder flow.

results displayed are at $t = 40k = 1.846$, when X = displacement of the piston = 0.3692. In Tables II and III we display the values of the horizontal and vertical velocity fields. In Fig. 4 we plot the vectors $(u - u_0, v)$, i.e., the difference between the flow with $\nu = 0$ and the flow with $\nu \neq 0$. The correct rotational behavior can be observed. In Fig. 5 we display the positions of the vortex centers in the lower half of the domain. At this t, there are 531 vortex sheets in the calculations, and the total computing time has been 9 minutes on a CDC 6400 computer. It must be pointed out that the boundary layer thickness is $O((\nu t)^{1/2})$; i.e., it varies from 0 to $O(\Delta)$, and that we are considering effects induced by the internal mechanics of the boundary layer, which would normally require a fine grid for adequate resolution.

CONCLUSION

We have presented a grid free method for studying boundary layers. The two main features of this method are: (i) the use of vortex sheet segments as computational elements, and (ii) a new method for generating vorticity at walls. It is expected that this algorithm will be mainly useful as a component of hybrid methods, and an example of such use has been given.

One can see that an algorithm based on non-rotating vortex sheets cannot reproduce the effects characteristic of turbulent boundary layers (see, e.g., [7]). Turbulence effects can conceivably be taken into account by replacing the molecular viscosity which determines the variance of the random variable η by an eddy viscosity. However, in later work we expect to use our present algorithm as a vorticity generation method for a hybrid method, in which the main part of the calculation will be carried out through the use of vortex elements of more elaborate structure; the sheets will be effectively relegated to the viscous sublayer.

It is obvious that a price must be paid for the removal of numerically induced viscosity in our method, and this price is statistical error. It is hoped that there will be a substantial number of applications in which such price is worth paying. It is also obvious that the present method generalizes trivially to three dimensional flows.

Note. The programs used to obtain the results above are available from the author.

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